STRESS AND STRAIN – A MATHEMATICAL TREATMENT

Structural engineers need to master this stuff to stress-test bridge design. Structural geologists need to master this stuff to stress-test deformed rocks to define controls on ore location.

Standard symbols that I will use

F **FORCE**
The energy input to a stressed rock.

σ **STRESS** (Greek letter sigma)
Stress is defined as force per unit area. You never see stress: you can only see its effects.

ϵ **STRAIN or DEFORMATION** (Greek letter epsilon)
Strain is what you see in a deformed rock. It is the combined result of all the stresses it has experienced over time.

τ **SHEAR STRESS** (Greek letter tau)
Shear stress causes strain such that *two rock masses move laterally past each other across a shear surface.*

α (Greek letter alpha) the angle between a plane and the principal stress axis
For a definition of the stress axes, see below.

E **YOUNG'S MODULUS**
A measure of the "stiffness" of a rock. Its resistance to strain under elastic conditions.

V **POISSON'S RATIO** (Greek letter nu)
The ratio of axial strain to lateral strain (quantifies how a material deforms in a direction perpendicular to an applied force).

m **POISSON'S NUMBER**
The reciprocal of Poisson's ratio.

G **MODULUS OF RIGIDITY**
Shear stress divided by strain: τ/ϵ

S **COHESIVE STRENGTH**

T **TENSILE STRENGTH**

The Principal Stress Axes

On any volume of rock, at any given moment, forces with a range of magnitude act in all possible directions. However, as opposed forces will cancel or reinforce each other, the infinity of forces can be resolved mathematically into just three principal forces at right angles (i.e. orthogonal) to each other. If we consider a unit cube of rock, then the force acting on each the three paired opposed faces of the cube are known as the ***principal stress directions or stress axes***. Stressed rocks behave as though these three orthogonal forces were acting on them.

In the general case, known as ***deviatoric stress***, the three principal stress axes have different magnitudes and are designated σ_1 for the direction of greatest stress, σ_3 for the direction of least stress and σ_2 for the stress direction which has a value that is intermediate between the other two.

The special case, where stress has the same magnitude in all directions, is known as ***static or non-deviatoric stress***.

Deformation (strain) in a rock is caused by the difference in magnitude between the principal stresses acting on it. The greater the difference, the more likely the rock is to deform. Failure is most likely to occur along surfaces that lie between the greatest (σ_1) and the least (σ_3) stress axes. Thus, when describing the relationship between stress and deformation, we can simplify our analysis by considering only those planes which lie at an angle between these to these two axes, and contain the σ_2 axis. This is known as ***plane (or monoclinic) strain.***

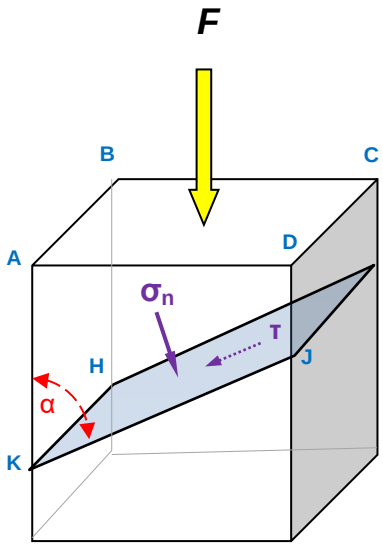
Plane strain is a useful and simplifying assumption. Although it is never wise to completely ignore the third dimension, as a first approximation, most deformed rocks, most of the time, can be usefully analysed using a plane strain model. It allows graphical illustration by means of a *2-D* section at right angles to σ_2. On an orthogonal *2D* section through a *3D* object, lines represent planes oriented at right angles to the paper. Points represent lines at right angles to the paper.

A Mathematical Treatment of Stress and Strain Answers Two Key Questions

Given the values of the principal stress axes, what are the resolved stresses acting on a plane at any given angle to those stress axes?

Given the stresses acting on that plane, at what point, and in what manner, will it deform?

The maths we will employ are basic trigonometry and algebra.



In the diagram, a Force (***F***) acts on face ***ABCD*** of a unit cube of rock.

Since force is defined in terms of Force per unit area, the stress on this face is σ_1

The rock contains an imaginary plane ***HIJK*** at an angle α (alpha) to ***F***.

Knowing the value of σ_1, how do we determine σ_n – the stress acting *normal* to ***HIJK***?

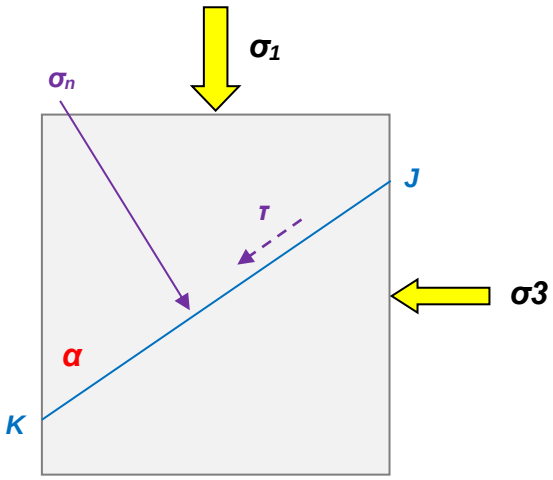
How do we determine τ *(tau)* **–** the shear stress acting *along* ***HIJK***?

Let the area of ***ABCD*** = ***a;*** let the area of ***HIJK*** = a^l***;*** let σ_n represent stress normal to ***HIJK***.

By simple trigonometry: $a^l = a / \sin \alpha$

F_n is the resolved force acting on ***HIJK.*** It is equal to $F\sin\alpha$
σ_n is the resolved stress normal to ***HIJK***
Therefore: $\sigma_n = F\sin\alpha / a^l$
Substituting for a^l : $\sigma_n = F\sin\alpha / (a/\sin\alpha)$
Rearranging terms: $\sigma_n = \sigma_1 \sin^2\alpha$ (***Equation 1)***

Plane Strain Graphic:



JK is a section through a plane at an angle of α from the principal stress direction σ_1.

Given the values of σ_1 and σ_3, how do we calculate the stresses on plane JK?

Stress on JK can be resolved into two components:
stress normal to JK designated by σ_n, and..

Stresses along JK – designated by tau (τ)

σ_n = resolved σ_1 stress + resolved σ_3 stress.
Re-arranging terms from ***Equation 1***, above:

$$\sigma_n = \sigma_1 \sin^2\sigma + \sigma_3 \sin^2\alpha$$

and, by similar algebraic re arrangement:

$$\tau = \sigma_1 \cos\alpha . \sin\alpha - \sigma_3 \cos(90-\alpha) . \sin(90-\alpha)$$

or:

$$\tau = (\sigma_1 - \sigma_3) \sin\alpha . \cos\alpha$$

It is useful (as will become apparent) to write the above equations in terms of the double angle 2α. This takes a bit of manipulation, but here it is.

$$\sigma_n = \frac{\sigma_1 + \sigma_3}{2} - \frac{\sigma_1 - \sigma_3}{2} . \cos 2\alpha \qquad \textit{(Equation 2)}$$

and..

$$\tau = \sigma_1 - \sigma_3 . \sin 2\alpha \qquad \textit{(Equation 3)}$$

Equations ***2*** *and* ***3*** allow calculation of the amount of resolved pure shear stress (***σ_n***) and simple shear stress (tau, ***τ***) across any plane that lies at a specified angle (***α***) to a principal stress direction (***σ***).

There is a graphical way in which these equations can be visualised…

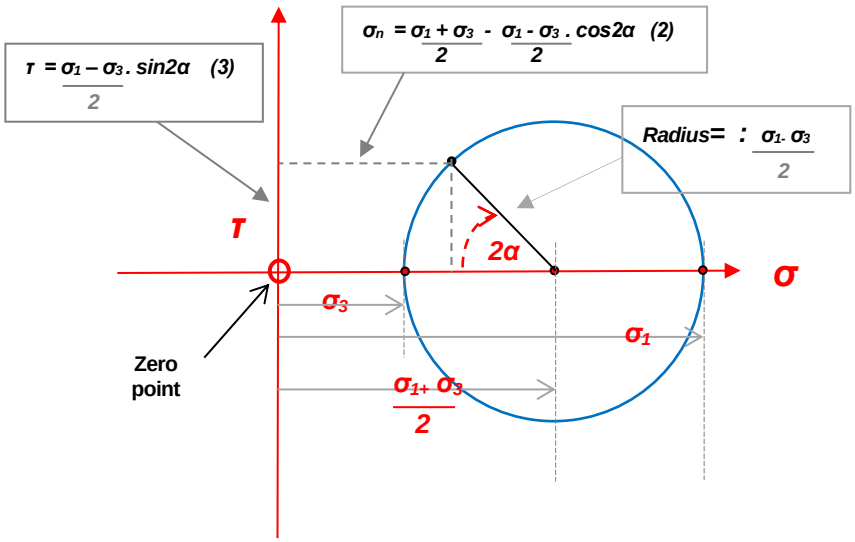
The Mohr Diagram

In 1885, German structural engineer Otto Mohr devised a graphical way of plotting equations 2 & 3. Known eponymously as the **Mohr Diagram,** it a simple 2-axis graph where ***τ*** (Greek letter *tau* - simple shear) forms the vertical axis and ***σ*** (pure shear) the horizontal axis.

Now we can see why Mohr formulated equations *2* and *3* the way he did. Any circle drawn on the Mohr diagram centred on the point : $\frac{\sigma_1 + \sigma_3}{2}$

and with a radius of : $\frac{\sigma_1 - \sigma_3}{2}$ will satisfy equations *2 & 3*.

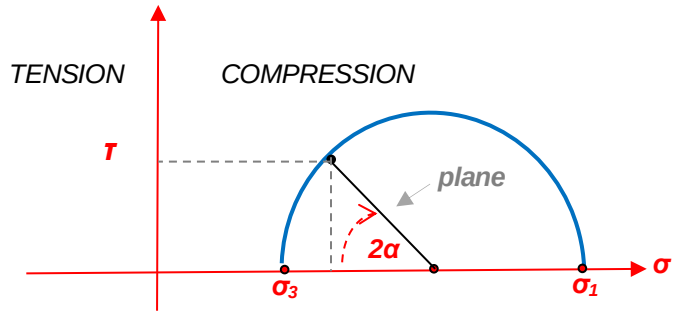
The stress difference $\sigma_1 . \sigma_3$ thus defines the radius of the circle. Any radius is a potential deformation plane and can be specified by the angle 2α that it makes with the α axis. The point at which a radius intersects the Mohr circle allows the amount and type of brittle deformation for that potential deformation plane to be read off from the two axes.



Here is how it works: Any given stress difference ($\sigma_1 - \sigma_3$) is drawn as a circle on the diagram. This is the Mohr circle. A plane with an angle of 2α to σ1 is drawn as a radius to the circle.

The coordinates of the point where the radius cuts the circle circumference allows the simple shear tau (τ) and pure shear (σ_n) stresses on the plane to be read off along the appropriate axes of the graph.

THE MOHR GRAPHICAL REPRESENTATION OF STRESS AND STRAIN RELATIONSHIPS

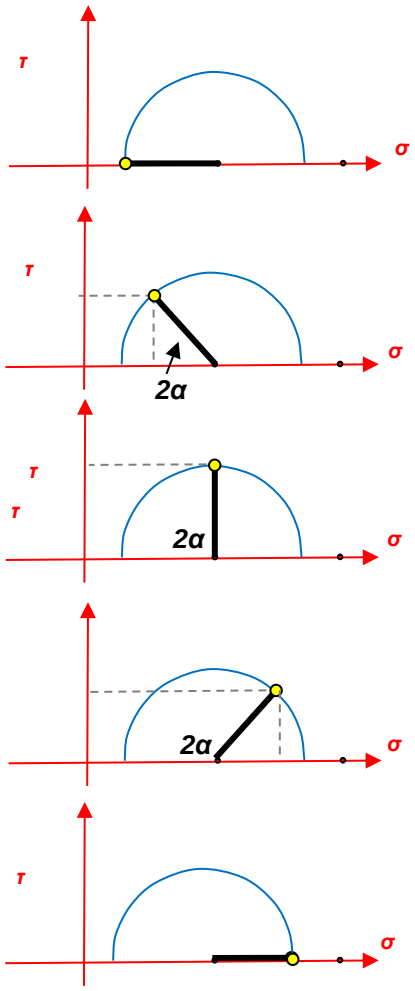


Since negative shear stress (tau or τ) has no physical reality, the lower half of the diagram (below the sigma axis) can be dispensed with.

The left of the diagram, where σ is negative, corresponds to states of tension. The right of the diagram, where σ is positive, corresponds to states of compression

Application of the Mohr Diagram

The Mohr diagram graphically illustrates how the stresses across a plane vary with the angle *2α* that it makes with the principal stress axis.



A Where α is 0, simple shear is zero and pure shear is at minimum – equal to σ_3

B With increasing α between 0 and 45°, both pure shear and simple shear steadily increase

C When α reaches 45°, simple shear is at its maximum possible value, but pure shear is still high.

D With any increase in α over 45°, simple shear along the potential failure plane will decrease. The amount of pure shear across the plane continues to rise.

E When α reaches 90°, simple shear is zero. Pure shear across the plane has reached its maximum, equal to σ_1

Elasticity

Elasticity is the deformation (strain) that takes place before failure such that, if the stress is removed, the strain is also removed

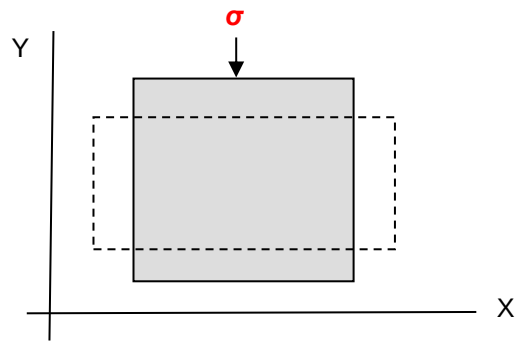
Rocks at surface may not seem very elastic (they're not) but with increasing temperature and confining pressure (lithostatic and hydrostatic stress, see below) they become increasingly elastic.

The stress / strain relationship under elastic conditions is linear
This is known as ***Hooke's Law***

The relationship is given by a constant called ***Young's Modulus,*** designated by the letter symbol ***E***

E = Stress / Strain = σ / ϵ *(where ϵ - epsilon - is the symbol for strain)*

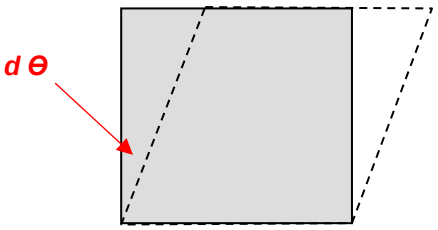
For any plane in XY space subject to pure shear:



V = Change in length of Y / Change in length of X ***Poisson's Ratio***

(V is Greek letter nu)

For any plane in XY space under simple shear:

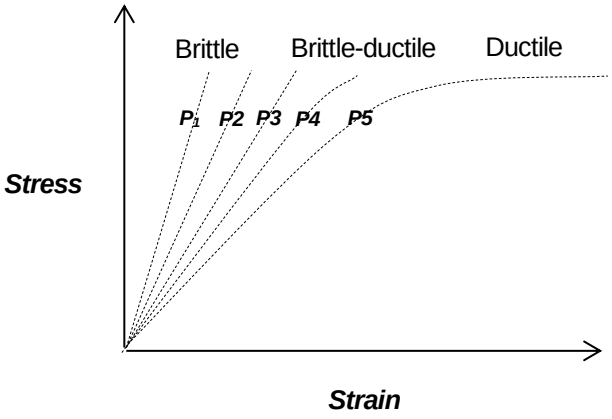


Shear Stress / Shear Strain = τ / d Θ ***Modulus of Rigidity***

Brittle failure

A rock deformed past its elastic limit will fracture in a brittle way. Stress/strain relationships in brittle fracture, like those in elastic deformation, are linear.

With increasing confining pressure and temperature, ductile deformation mechanisms come into play and eventually predominate over brittle deformation. Ductile deformation is non-linear.



Lines ***P1*** – ***P5*** represent deformation pathways ranging from brittle to ductile with increasing confining pressure and temperature.

Confining pressure is defined as lithostatic stress minus hydrostatic stress (see below)

Compressive and Tensional Static Stresses (σ_z) in the Earth's Crust

Static stresses in the earth's crust are the result of gravity. They are called *"Static"* because the magnitude of the stress is equal in all directions.

There are two kinds of Static stress in the earth's crust:

Lithostatic stress is caused by gravitational loading on rock masses and is always positive - that is, it tends to compress the rock. Lithostatic stress increases with depth in a linear manner, described mathematically thus:

$\sigma_z = pm.g.d$ where: ***pm*** is rock density
g is the gravity constant
d is depth below surface

Hydrostatic stress is caused by pressure of fluids (usually water) contained in the fractures, micro-fractures and intergranular spaces of the rock.

The hydrostatic pressure of pore fluids acts to *expand* the rock volume and is thus always negative. It acts counter to the compressive effects of lithostatic pressure.

If pore fluids are in continuous contact through to the earth's surface their pressure is equal to the hydraulic head and will increase in a linear manner with increasing depth.

Both lithostatic *(+ve*) and hydrostatic (*-ve*) pressure increase with depth. Since lithostatic pressure is almost always greater than hydrostatic pressure (because rock is denser than water) the total compressive force on the rock will normally increase with depth. A special case exists where pore fluids are sealed from the surface by an impermeable layer. This layer is known as an ***aquiclude*** if it forms a complete and permanent seal; and an ***aquitard*** if the seal is partial or intermittent. In either circumstance, local tectonic stress may cause the *hydrostatic pressure* to become *greater than lithostatic pressure.* This condition known as ***over pressuring***.

Over pressured pore fluids can move rock from a compressive to a tensional strain state causing the rock to fail by tensional fracturing, even at great depths.

Epigenetic metal deposits are normally hosted in tensional veins or in zones relatively low compressive pressure. It is thus important to consider the circumstances in which such conditions might apply.

To summarise: rocks in the crust are subject to:

Tectonic or deviatoric stress (stresses of different magnitude in different directions). Tectonic stress may be positive (compressive) or negative (tensional). Negative stresses are at their maximum in the upper few kilometres of the crust.

Compressive lithostatic stress, increasing with depth.

Tensional hydrostatic stress. These stresses are at maximum if pore fluids in the rock are overpressured as a result of tectonic stress.

Tensional fractures may form:

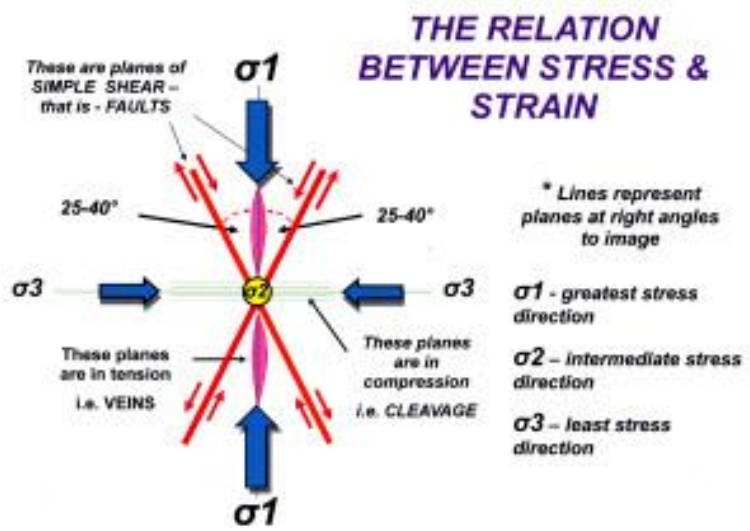
In extensional tectonic regimes in the top few kilometres of the crust where brittle fracturing is the dominant deformation style. This is the typical environment of **epithermal veins**.

At any crustal depth if pore fluids are sufficiently over pressured. This is the typical environment of **mesothermal epigenetic veins**

Navier Coulomb Criteria for Brittle Fracture

You might expect from the mathematical treatment above that simple shear fractures (faults) will occur along planes oriented at *45°* to the principal stress direction since that is the plane along which we have shown there is a *maximum resolved simple shear stress* (see the Mohr diagram *C*, above). But maximum shear stress only results in maximum shear strain in perfectly elastic mediums, and no rocks are perfectly elastic. Rather, rocks possess, to differing degrees, an *internal strength* that resists fracture. This causes shear fractures to occur at alpha angles that are less than *45°*. Why? Because at *α=45°*, the resolved stress *normal* to the plane (σ_n) is also high and will inhibit movement along that plane. A lower *α* angle has a lower σ_n value, thus enabling a shear fracture to form.

In a strong and competent rock (igneous rocks, quarzites etc..), brittle shears might form at angles of *25°- 30°* to the principal stress axis. In a much weaker rock (shale, mudstone etc..) faults that form will be along planes at *30°- 40°* to the principal stress axis.



The difference in alpha angle between the plane of maximum resolved simple shear and the ***actual*** fracture plane that develops is known as the ***Angle of Internal Friction,*** and is denoted by the symbol Φ_i **.**
The ***Coefficient of Internal Friction -*** u_i - is the ***tan*** of the Angle of Internal Friction.

Simple Shear Fracture May Take Place Along a Surface When:

Shear stress on surface ≥ Cohesive Strength (***S***) + Frictional Resistance to movement.

Frictional Resistance = Stress normal to surface (σ_n) + Coefficient of Internal Friction (u_i)

Failure can occur when:

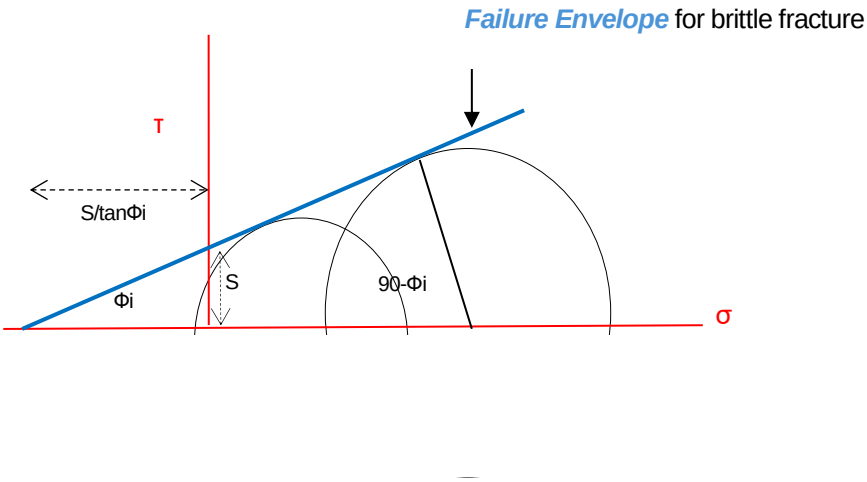
$$\tau = S + \sigma_n u_i \qquad (\tau - \textit{tau is simple shear stress along the plane})$$

or

$$\tau = S + \sigma_n \tan \Phi i$$

These are linear equations and can be shown on the Mohr diagram as a straight line. In the figure below, in any given stressed rock, various potential stress states (***σ1– σ3***) are represented by a series of circles. A single line tangent to these circles then represents the point at which that rock will fail under brittle conditions and compressive stress. The failure line (shown in blue) is known as the ***Failure Envelope***.

Only two stress states are shown on the diagram, but they sample of an infinite series of such circles.



This failure envelope model illustrates the ***Navier Coulomb*** brittle failure criteria. The criteria work well at representing simple shear deformation. This is the deformation that takes place in the right-hand field of the Mohr diagram. In the tension field (left half of the graph) deformation does not primarily take place by simple shear fractures but by extension fractures or a combination of tension fractures and shears. Thus, on the left of the graph, a different set of equations are necessary to define the failure envelope. These equations are known as the Griffin Criteria of Brittle Fracture.

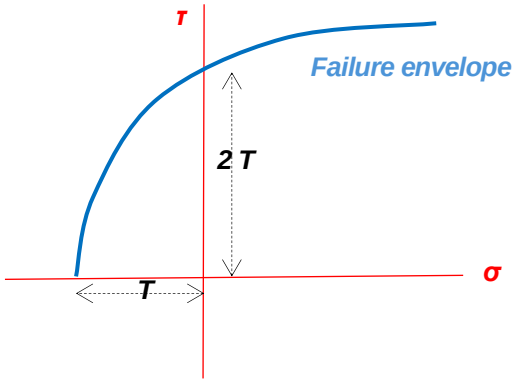
Griffith Criteria of Brittle Fracture

Most rocks contain fractures. They occur at all scales. Stress that is higher than the average for the rock will localise there. The effect is greatest under tensional conditions because, under compression, cracks at a high angle to the compressive force (σ^1) will tend to close.

According to the Griffith Criteria, in the tensional field, relationships between the principal stresses at failure are:

$\tau^2 + 4T\sigma_n - 4T^2 = 0$ (*Where* ***T*** *is the tensile strength of the rock and tau* τ *is the shear stress*)

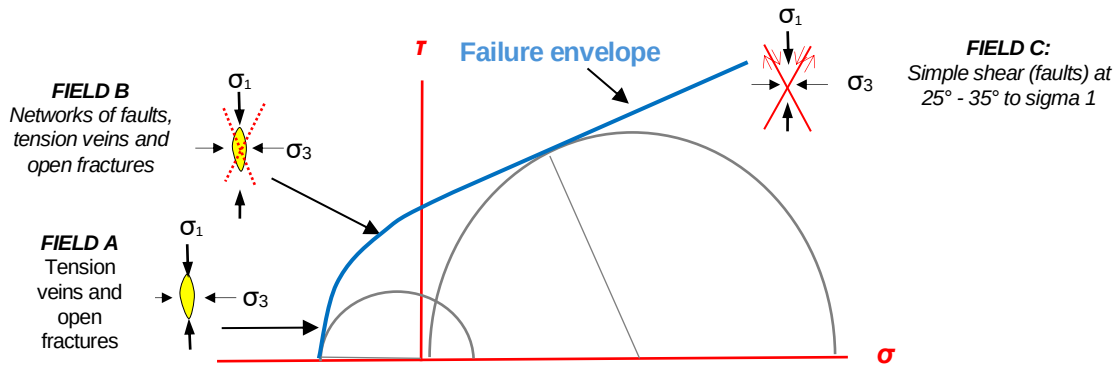
This is a quadratic equation and will plot on the Mohr graph as a parabola:



The Griffith criteria match experimental data for rocks deforming in tensional stress environments.

The Grand Synthesis: Putting it All Together

For a full description of the failure envelope for a variety of rocks, under a wide range of stress conditions, we need to combine the **Griffith** *and* the **Navier-Coulomb** failure criteria.



Further Reading

Ramsey J.G. 1967. *Folding and Fracturing of Rocks* (McGraw Hill, 1967. Library of Congress No. 67-14897).
A classic. Still the best treatment of this subject. Easily obtained on the internet.

Ramsey J.G. & Huber M.I. 1987: *The Techniques of Modern Structural Geology.* Academic Press, 700p

Elasticity, Fracture and Flow by J.C. Jaeger. (Methuen,1969. ISBN 416 14880 8).
A more detailed treatment that expands the topic to stress and strain in 3D and 4D applications (*the 4th dimension is time: but the math is more advanced: beyond my pay grade*).

R. W. Marjoribanks. Previous blog posts under the category heading ***Structural Geology.*** *These posts are designed to give practical field applications of stress – strain relationships that are of value to the explorer for metals.*

Any good structural geology textbook (post-Ramsey) will cover the topic of this post. But structural geology textbooks are generally written by academics who, in most cases, have little experience in economic geology. In this post I have attempted to remedy that by showing how the basic principles of rock mechanics can be applied to understanding the structural controls on epigenetic metal accumulation.