STRESS AND STRAIN – A MATHEMATICAL TREATMENT

Structural engineers need to master this stuff to stress-test bridge design. Structural geologists need to master this stuff to stress-test deformed rocks to define controls on ore location.

Standard Mathematical Synbols

F **FORCE**
The energy input to a stressed rock.

σ (sigma) **STRESS** (often called Pure Shear).
Stress is defined as force per unit area. You never see stress: you can only see its effects.

ϵ (epsilon) **STRAIN**
Strain is what you see in a deformed rock. It is the combined result of all the stresses it has experienced over time.

τ (tau) **SHEAR STRESS** (often called Simple Shear)
Simple shear is when two rock masses move laterally past each other across a shear surface.

α (alpha) the angle between a plane and the principal stress axis
For a definition of the stress axes, see below.

E **YOUNG'S MODULUS**
A measure of the "stiffness"of a rock. Its resistance to strain under elastic conditions.

V (nu) **POISSON'S RATIO**
The ratio of axial strain to lateral strain (quantifies how a material deforms perpendicularly to an applied force).

m **POISSON'S NUMBER**
The reciprocal of Poisson's ratio.

G **MODULUS OF RIGIDITY**
Shear stress divided by strain: τ/ϵ

S **COHESIVE STRENGTH**

The Principal Stress Axes

At every point in a rock, at any given moment, forces with a range of magnitude act in all possible directions. However, as opposite forces will either tend to cancel or reinforce each other, the infinity of forces can be resolved mathematically into just three principal forces at right angles (orthogonally) to each other. If we consider a unit cube of rock, then the force acting on each the three opposed faces of the cube are known as the ***principal stress directions or stress axes***. Stressed rocks (which know no maths at all) behave as though just these three orthogonal forces were acting on them.

In the general case, known as ***deviatoric stress***, the three principal stress axes have different magnitudes and are designated (σ_1) for the direction of greatest stress, (σ_3) for the direction of least stress and (σ_2) for the stress direction which has a value that is intermediate between the other two.

The special case, where stress has the same magnitude in all directions, is known as ***static or non-deviatoric stress***.

Deormation in a rock is caused by the difference in magnitude between the principal stresses acting on it. The greater the difference, the more likely the rock is to deform. Failure is thus most likely along surfaces that lie between the greatest (σ_1) and the least (σ_3) stress axes. Thus, when describing the relationship between stress and deformation, we can therefore simplify our analysis by considering only those mathematical planes that lie at an angle between these to these two axes, and contain, or are parallel to, the σ_2 axis. This situation is known as ***plane (or monoclinic) strain.***

Plane strain is a useful and simplifying assumption. Although it is never wise to completely ignore the third dimension, as a first approximation, most deformed rocks, most of the time, can be usefully analysed using a plane strain model. It allows graphical illustration by means of a *2-D* section at right angles to σ_2. On a *2D* section through a *3D* object, lines represent planes oriented at right angles to the paper.

A Mathematical Treatment of Stress and Strain Answers Two Key Questions

> Q1: Given the values of the principal stress axes, what is the stress acting on a plane at any given angle to those stress axes?
>
> Q2: Given the stresses acting on a plane, at what point, and in what manner, will it deform?

The maths we will employ are basic trigonometry and algebra. Here goes...

In the diagram below, a Force (***F***) acts on a face (labelled ***ABCD***) of a unit cube of rock.

By definition, the stress on the face is σ_1

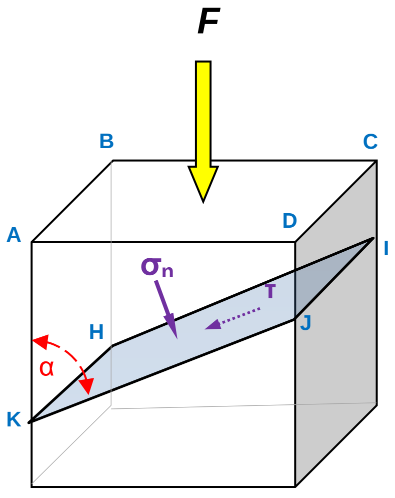
The rock contains an imaginary (or mathematical) plane (***HIJK***) that lies at an angle, α , to ***F***. Knowing the value of σ_1, how do we determine σ_n – the stress acting normal to ***HIJK*** ?

Let area ABCD = a

Let area HIJK = a'

Therefore $a' = a/\sin\alpha$

Since stress (σ) = Force per unit area σ_1 (the stress normal to ABCD) = F/a



Fn **– the resolved force normal to HIJK**

σ_n - the stress normal to HIJK

Now, Fn = F sinα, therefore:

$$\sigma_n = \frac{F\sin\alpha}{a'}$$

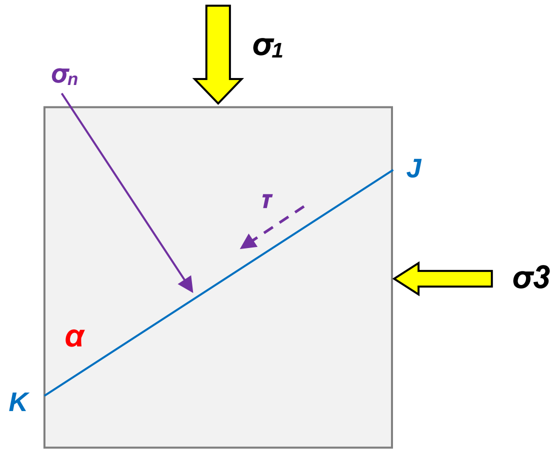
Substituting for a':

$$\sigma_n = \frac{F\sin\alpha}{(a/\sin\alpha)}$$

Re-arranging the terms:

$$\sigma_n = \sigma_1 \sin^2\alpha \quad (1)$$

Considering only strain in the plane of *σ1* and *σ3*: (plane strain)



JK is an imaginary mathematical plane that lies at an angle of α from the principal stress direction σ_1.

Given the values of σ_1 and σ_3, how do we calculate the stresses on plane JK?

Stresses on JK can be resolved into two components:
stress at right angles or normal to JK (pure shear, designated σ_n) and

simple shear stresses along JK – designated by the letter tau (τ)

σ_n = resolved σ_1 stress + resolved σ_3 stress.
Re-arranging from equation 1, above:

$$\sigma_n = \sigma_1 \sin^2\sigma + \sigma_3 \sin^2\alpha$$

and, by similar algebraic re arrangement:

$$\tau = \sigma_1 \cos\alpha.\sin\alpha - \sigma_3 \cos(90-\alpha).\sin(90-\alpha)$$

or:

$$\tau = (\sigma_1 - \sigma_3)\sin\alpha.\cos\alpha$$

It is useful (as will become apparent) to write the above equations in terms of the double angle 2α. This takes a bit of manipulation, but here it is.

$$\sigma_n = \frac{\sigma_1 + \sigma_3}{2} - \frac{\sigma_1 - \sigma_3}{2} . \cos 2\alpha \qquad (2)$$

and

$$\tau = \sigma_1 - \sigma_3 . \sin 2\alpha \qquad (3)$$

What use are equations ***(2)*** & ***(3)*** ?

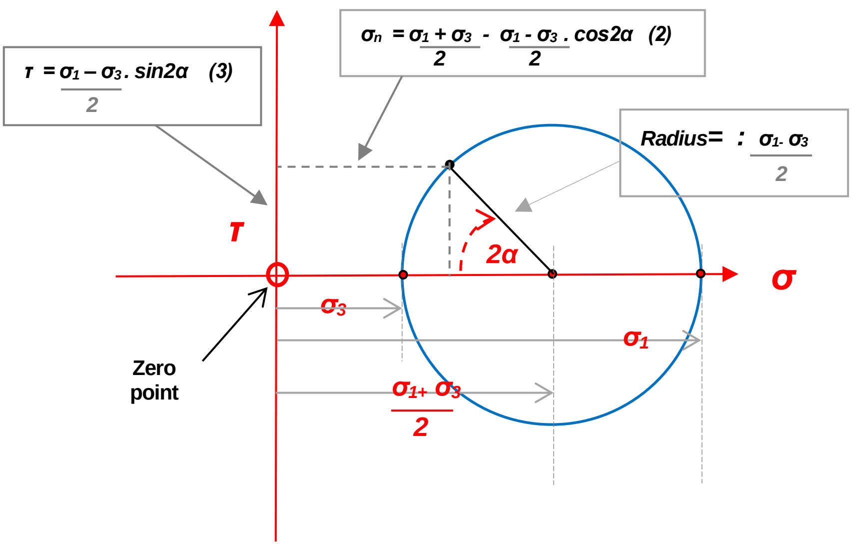
They allow calculation of the amount of resolved pure shear stress (***σ_n***) and simple shear stress (***τ***) across any plane that lies at a specified angle (***α***) to any principal stress direction (***σ***).

The Mohr Diagram

In 1885, German structural engineer Otto Mohr devised a graphical way of plotting equations (2) & (3). Known eponymously as the **Mohr Diagram,** it a simple 2-axis graph where τ (simple shear) forms the vertical axis and σ (pure shear) the horizontal axis. Now we can see why Mohr formulated equations *(2)* and *(3)* the way he did. Any circle drawn on the Mohr diagram centred on the point $\frac{\sigma_1 + \sigma_3}{2}$

and with a radius of $\frac{\sigma_1 - \sigma_3}{2}$ will satisfy the equations.

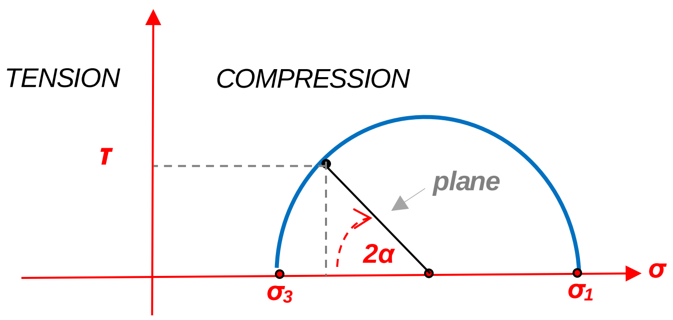
On the diagram, the radius of the circle is defined by the angle 2α it makes with the horizontal or σ axis. For any given principal stress difference, the Mohr diagram graphically shows the amount and type of brittle deformation to be expected on a plane making any specified angle (α) with the principal stress axis.



Here is how it works: Any given stress difference ($\sigma_1 - \sigma_3$) is drawn as a circle on the diagram. This is the Mohr circle. A plane with an angle of 2α to σ1 is drawn as a radius to the circle.

The coordinates of the point where the radius cuts the circle circumference allows the simple shear (τ) and pure shear (σ_n) stresses on the plane to be read off along the appropriate axes of the graph.

THE MOHR GRAPHICAL REPRESENTATION OF STRESS AND STRAIN RELATIONSHIPS

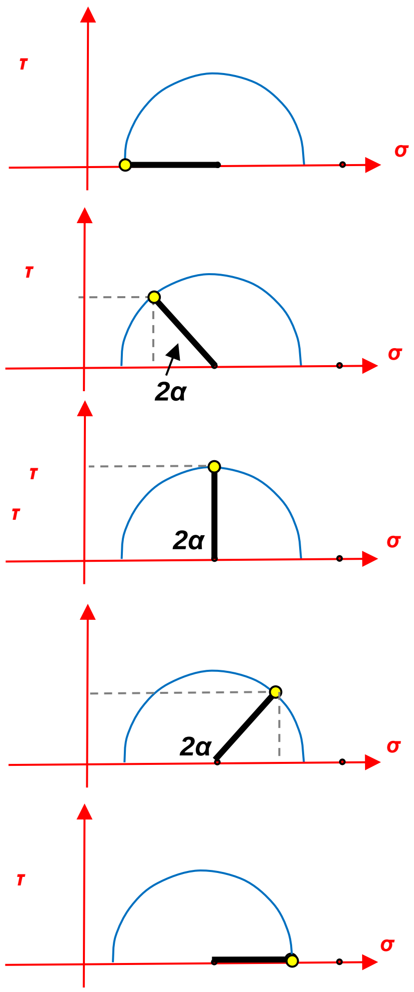


Since negative shear stress (tau or τ) has no physical reality, the lower half of the diagram (below the sigma axis) can be dispensed with.

The left of the diagram, where σ is negative, corresponds to states of tension. The right of the diagram, where σ is positive, corresponds to states of compression

Applications of the Mohr Diagram

The Mohr diagram graphically illustrates how the stresses across a plane vary with the angle (*2 alpha*) that it makes with the principal stress axis.



A Where α is 0, simple shear is zero and pure shear is at minimum – equal to σ_3

B With increasing α between 0 and 45°, both pure shear and simple shear steadily increase

C When α reaches 45°, simple shear is at its maximum possible value.

D With any increase in α over 45°, simple shear along the potential failure plane will decrease, but the amount of pure shear across the plane continues to rise.

E When α reaches 90°, simple shear is zero, but the pure shear component across the plane has reached its maximum and is equal to σ_1

Elasticity

Elasticity is the deformation (strain) that takes place before failure such that, if the stress is removed, the strain is also removed.

Rocks at surface may not seem very elastic (they're not) but with increasing temperature and confining pressure (lithostatic and hydrostatic stress, see below) they become increasingly elastic.

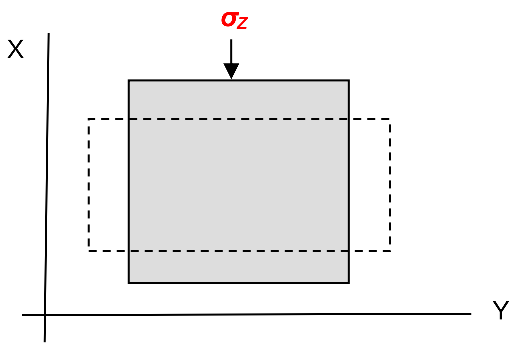
The stress / strain relationship under elastic conditions is linear
This is known as ***Hooke's Law***

The relationship is given by a constant called ***Young's Modulus*** and is designated by letter symbol ***E***

E = Stress / Strain = σ / ϵ *(where ϵ - epsilon - is the symbol for strain)*

$$E = \frac{\text{Stress}}{\text{Strain}} = \sigma / \epsilon$$

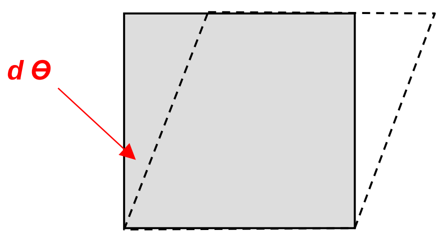
For any plane in XY space subject to pure shear stress:



Poisson's Ratio (V) = Change in the length of Z / Change in length of Y

$$V = \frac{\text{Change in the length of } Z}{\text{Change in length of } Y}$$

For any plane in XY space under simple shear stress:



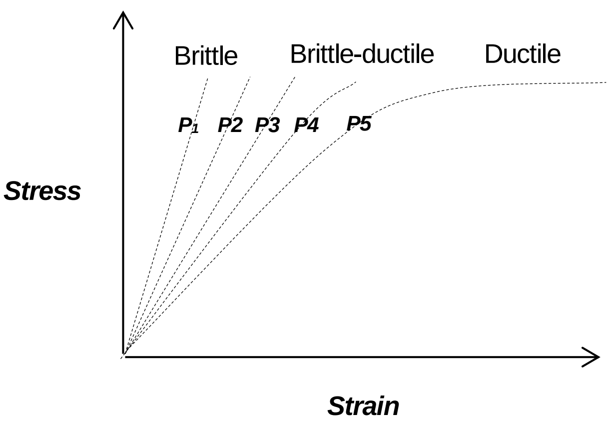
$$\frac{\textit{Shear Stress}}{\textit{Shear Strain}} = \frac{\tau}{d\theta}$$

Modulus of Rigidity

Brittle failure

When a rock has deformed to its elastic limit, it will fracture in a brittle way. Stress/strain relationships in brittle fracture, kike those in elastic deformation, are linear.

With increasing confining pressure and temperature, ductile deformation mechanisms come into play and eventually predominate over brittle deformation. *Ductile deformation is non-linear.*



P1 – P5 represents deformation pathways from brittle to ductile strain with increasing confining pressure and temperature.

Confining pressure is defined as lithostatic stress minus hydrostatic stress (see below)

Compressive and Tensional Static Stresses (σ_z) in the Earth's Crust

Static stress is the result of gravity. Called "Static" because the magnitude of the stress is equal in all directions.

There are two kinds of Static stress:

Lithostatic stress is caused by gravitational loading on rock masses and is always positive - that is, it tends to compress the rock. Lithostatic stress *increases with depth in a linear way*:

$\sigma_z = pm.g.d$ where:

pm	is rock density
g	is the gravity constant
d	is depth below surface

Hydrostatic stress is caused by pressure of fluids (usually water) contained in the fractures, micro-fractures and intergranular spaces of the rock.

The internal pressure of pore fluids tries to *expand* the rock volume. Hydrostatic pressure is thus always negative. It acts counter to the compressive effects of lithostatic pressure.

If pore fluids are in continuous contact through to the earth's surface their pressure is equal to the hydraulic head and will increase in a linear manner with increasing depth.

Both lithostatic *(+ve*) and hydrostatic (-*ve*) pressures increase with depth. Since lithostatic pressure is greater than hydrostatic pressure (because rock is denser than water) the total compressive force on the rock will normally increases with depth. Sometimes, however, pore fluids are sealed from the surface by an impermeable layer. This layer is known as an ***aquiclude*** if it forms a complete and permanent seal; and an ***aquitard*** if the seal is partial or intermittent. In either circumstance, local tectonic stress, may cause the *hydrostatic pressure to become greater than lithostatic pressure*, a condition known as ***over pressuring***.

Over pressured pore fluids can move rock from a compressive to a tensional strain state causing the rock to fail by tensional fracturing, even at great depths.

Epigenetic deposits are normally hosted in tensional veins or zones so it is important to consider the circumstances in which these conditions might apply.

To summarise: rocks in the crust are subject to:

Tectonic or deviatoric stress (stresses of different magnitude in different directions). Tectonic stress may be positive (compressive) or negative (tensional). Negative stresses are at their maximum in the upper few kilometres of the crust.

Compressive lithostatic stress, increasing with depth.

Tensional hydrostatic stress. These stresses are at maximum if pore fluids in the rock are overpressured as a result of tectonic stress.

Tensional fractures may form:

In extensional tectonic regimes in the top few kilometres of the crust where brittle fracturing is the dominant deformation style. This is the typical environment of **epithermal veins**.

At any crustal depth if pore fluids are sufficiently over pressured. This is the typical environment of **mesothermal epigenetic veins**

Navier Coulomb Criteria of Brittle Fracture

You might expect from the mathematical treatment above that simple shear fractures (faults) will occur along planes oriented at *45°* to the principal stress direction, since that is the plane along which there is a *maximum resolved simple shear strain* (as illustrated in the Mohr diagram labelled ***C***, above). But that theoretical maximum only occurs in perfectly elastic mediums, and no rocks are perfectly elastic. Rather, rocks possess, to differing degrees, an *internal strength* that resists fracture. This causes shear fractures to occur at alpha angles less than *45°*. Why? Because at $\alpha = 45°$, the resolved stress *normal* to the plane (σ_n) is still high, and this inhibits shear movement along that plane. A lower α angle has a lower σ_n value, thus enabling a shear fracture to form.

Thus, in a strong and competent rock (igneous rocks, quarzites etc..), shears might form along planes at angles of *25°- 30°* to the principal stress axis. In a much weaker rock (shale, mudstone etc..) faults that form will be along planes at *30°- 40°* to the principal stress axis.

The difference in alpha angle between the plane of maximum resolved simple shear and the ***actual*** fracture plane that develops is known as the ***Angle of Internal Friction,*** and is denoted by the symbol Φ_i **.**

The ***Coefficient of Internal Friction -*** u_i - is the ***tan*** of the Angle of Internal Friction.

Simple Shear Fracture May Take Place Along a Surface When:

Shear stress on surface ≥ Cohesive Strength (***S***) + Frictional Resistance to movement.

Frictional Resistance = Stress normal to surface (σ_n) + Coefficient of Internal Friction (u_i)

Failure can occur when:

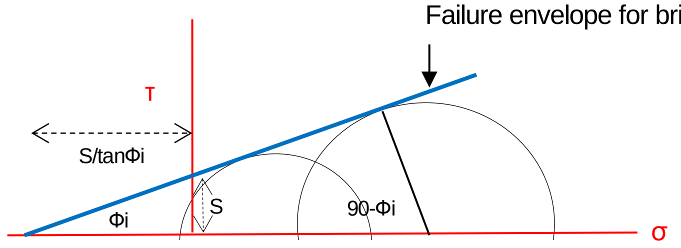
$$\tau = S + \sigma_n u_i$$

or

$$\tau = S + \sigma_n \tan \Phi i$$

These equations can be shown on the Mohr diagram. In the figure below, in any given stressed rock, various potential stress states ($\sigma 1 - \sigma 3$) are represented by a series of circles. A single line tangent to these circles then represents the point at which that rock will fail under these conditions. The failure line (shown in blue) is known as the ***Failure Envelope***.

Only two stress states are shown on the diagram below, but they represent an infinite series of such circles.



This failure envelope model illustrates the ***Navier Coulomb*** brittle failure criteria. The criterion works well at representing deformation involving compressive simple shear. This is the deformation that takes place in the upper right field of the Mohr diagram. However, in the tension field (upper left of the graph) deformation does not primarily take place by simple shear fractures (faults) but by extension fractures normal to σ3, or a combination of tension fractures and shears. Thus, on the left of the graph a different set of equations are necessary to define the failure envelope. These equations are known as the Griffin Criteria of Brittle Fracture.

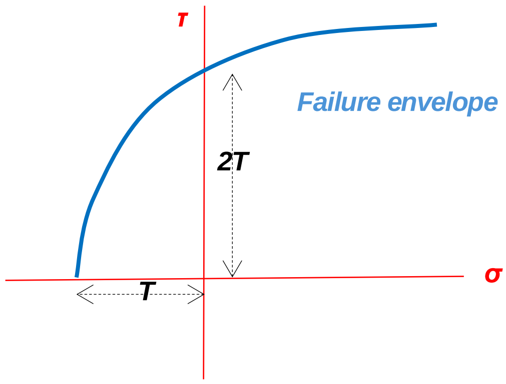
Griffith Criterion of Brittle Fracture

Rocks usually contain micro-fractures. Strain higher than the average for the rock as a whole tends to localise there, an effect is greatest under tensional conditions. Under compression, cracks at a high angle to the compressive force (sigma 1) will tend to close and the leveraging effects of micro fractures are reduced.

According to the Griffith Equation, in the tensional field, relationships between the principal stresses at failure are:

$$\tau^2 + 4T\sigma_n - 4T^2 = 0$$

Where ***T*** is the tensile strength of the rock. This is a quadratic equation which plots on the Mohr graph as a parabola, shown below:

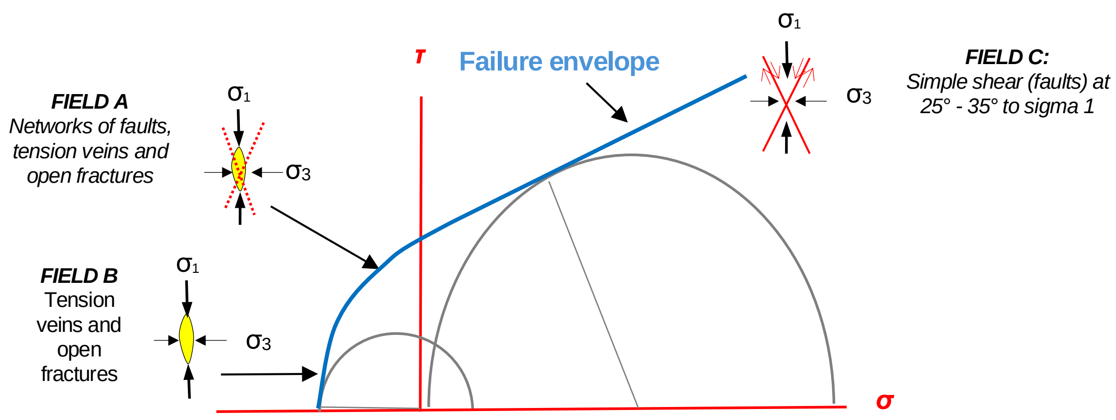


The Griffith criterion matches experimental data well for rocks deforming under tensional stress environments.

Putting it all together

For a full description of the failure envelope for a variety of rocks, under a wide range of stress conditions, we need to combine the **Griffith** *and* the **Navier-Coulomb** failure criteria.

Here it is, the grand synthesis:



Further Reading

The best treatment of this subject is still, in my opinion, John Ramsey's classic textbook: *Folding and Fracturing of Rocks* (McGraw Hill, 1967. Library of Congress No. 67-14897). Easily obtained on the internet.

For a more detailed treatment that expands the topic to stress and strain in 3D and 4D applications (*the 4th dimension is time: but the math is much heavier*), I recommend the book: *Elasticity, Fracture and Flow* by J.C. Jaeger. (Methuen,1969. ISBN 416 14880 8).

Any good structural geology textbook (post-Ramsey) will also cover these topics. But structural geology textbooks are generally written by academics with no particular interest or experience in economic geology. In this post I have attempted to remedy that by showing how the fundamental principals of rock mechanics can be applied to understanding the controls on epigenetic ore formation.